Heavy Quark $\bar{q}q$ Matrix Elements in the Nucleon from Perturbative QCD

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Abstract

The scalar heavy quark content of the nucleon, $\mathcal{M}_q = \langle N|m_q\bar{q}q|N\rangle$, is relevant for computing the interaction of dark matter candidates with ordinary matter, while $\mathcal{M}_s$ is important for predicting the properties of dense matter. We compute $\mathcal{M}_q$ in perturbative QCD to $\mathcal{O}(\alpha^3)$. As one goes from $\mathcal{M}_t$ to $\mathcal{M}_c$ the leading order contribution decreases as the number of light quarks is dropping, while the radiative corrections grow and are all positive. The leading source of uncertainty in the calculation is due to the poorly known value of $\mathcal{M}_s$. A related calculation suggests that a large value for $\mathcal{M}_s$ may be reasonable.

I. INTRODUCTION AND RESULTS

Knowledge of matrix elements $\mathcal{M}_q \equiv \langle N|m_q\bar{q}q|N\rangle$ provides us with some insight into the flavor properties of the nucleon wave function. Since scalars typically couple to $m_q\bar{q}q$, values of the quark mass operator matrix elements are also relevant for dark matter searches, as well as for experiments searching for new weak forces. The value of $\mathcal{M}_s$ is important for kaon condensation in dense nuclear matter [1], and its value has been the subject of a debate [2,3].

Heavy flavor $\mathcal{M}_q$'s have been calculated in perturbative QCD to the leading order in α [4]. Using a technique similar to the ones developed in [4–6] and making use of the four loop beta function, anomalous mass dimension, and three loop heavy flavor threshold matching coefficients available in the literature [7], in Section 2 we extend the calculation of heavy flavor $\mathcal{M}_q$'s to $\mathcal{O}(\alpha^3)$. We find

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$$\mathcal{M}_c = \frac{2}{27}(1 + 0.108 + 0.025 - 0.002)M(1 - x_{uds}) = 78.61(1 - x_{uds})\,\text{MeV}\,,$$

$$\mathcal{M}_b = \frac{2}{25}(1 + 0.071 + 0.009 - 0.002)M(1 - x_{uds}) = 80.92(1 - x_{uds})\,\text{MeV}\,,$$

$$\mathcal{M}_t = \frac{2}{23}(1 + 0.0402 + 0.0009 - 0.0007)M(1 - x_{uds}) = 84.89(1 - x_{uds})\,\text{MeV}\,, \tag{1}$$

where M is the nucleon mass (we use $M = 938.3$ MeV), and

$$x_{uds} = \frac{\Sigma_{\pi N} + \Delta_s M}{M}, \tag{2}$$

with $\Delta_s M$ being the shift in the nucleon mass which would result from setting $m_s = 0$,

$$\Delta_s M = \int_0^{m_s} dy \frac{\partial M}{\partial m_s}\bigg|_{m_s = y} = \int_0^{m_s} dy \, \langle N|\bar{s}s|N\rangle|_{m_s = y} \tag{3}$$

and

$$\Sigma_{\pi N} \equiv \frac{(m_u + m_d)}{2}\langle N|\bar{u}u + \bar{d}d|N\rangle \simeq 45\,\text{MeV}\,. \tag{4}$$

From (1) we see that there are two competing effects as one goes from the top to the charm quark matrix element: since the number of light quarks is dropping, the leading order contribution decreases, while the radiative corrections grow and are all positive. We also note that the greatest source of uncertainty in the above results is the poor knowledge of the strange matrix element entering $\Delta_s M$.

While the same technique is not applicable to $\mathcal{M}_s$ directly, since $m_s \sim \Lambda$, the scale where QCD becomes strong, in Section 3 we calculate the strange quark matrix element in the QCD with a hypothetical heavy strange quark and massless u and d quarks and find indications favoring a large value of $\mathcal{M}_s$ for the physical strange quark mass.

II. THE SIX FLAVOR CALCULATION

We consider six flavor QCD ($n_f = 6$) with three heavy quarks (c, b, t) and three light quarks (u, d, s). The running heavy quark masses are defined to be $\bar{m}_h(\mu)$ in the $\overline{\text{MS}}$ scheme, and we define the heavy quark mass m_h to be the "scale invariant mass", namely the solution to the equation $\bar{m}_h(m_h) = m_h$.

For $\mu > m_h$ the theory is described by the QCD Lagrangian with n_f "active" flavors; at the scale $\mu = m_h$ we integrate out the heavy quark h, and until the next heavy flavor threshold the effective theory consists of $n_f - 1$ flavor QCD, with a shifted gauge coupling, and a tower of nonrenormalizable interactions suppressed by powers of m_h^2. The $\mathcal{O}\left(1/m_h^2\right)$ contribution, for example, comes from the operator $(D_\mu G^{\mu\nu})^2/m_h^2$ in the effective theory [4]. The value of the coupling and of the coefficients of the nonrenormalizable interactions are fixed in perturbation theory by matching S-matrix elements in the full and effective theories. The effective theory is asymptotically free, and gets strong at the scale Λ.

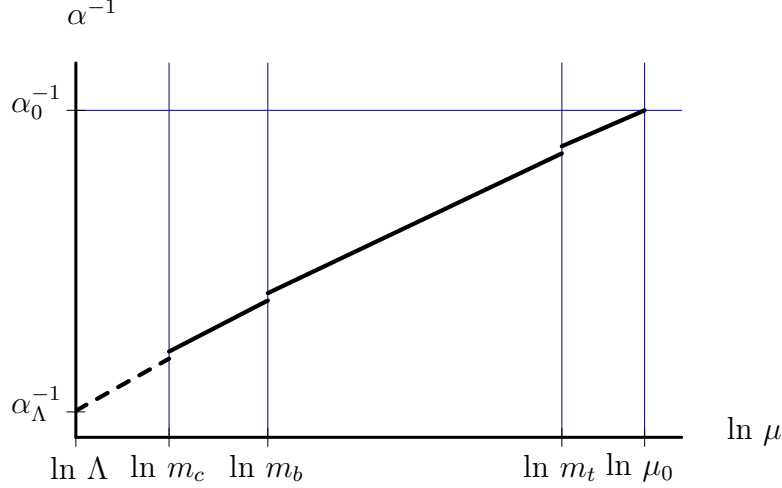The mass of the nucleon M is given by an expansion in m_h^{-2} and m_l of the form

FIG. 1. Shown here is a schematic plot of the QCD coupling $\alpha(\mu)^{-1}$ versus $\ln \mu$, in the $\overline{\text{MS}}$ scheme. The threshold corrections have been exaggerated for illustrative purposes. Keeping μ_0, α_0 and α_Λ fixed and varying $\bar{m}_q(\mu_0)$ one varies the value of Λ as well the locations of (some of) the heavy flavor thresholds $\bar{m}_h(m_h) = m_h$.

$$M = \Lambda \times \left[c_0 + \sum_{n=1}^{\infty} \sum_{h} c_{nh} \left(\frac{\Lambda^2}{m_h^2} \right)^n + \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \sum_{h,l} c_{nm,hl} \left(\frac{m_l}{\Lambda} \right)^m \left(\frac{\Lambda^2}{m_h^2} \right)^n \right] + \Delta_s M, \qquad (5)$$

where h is a generic index for the heavy quarks, l is a generic index for u and d, and $\Delta_s M$ is defined in (3). The expansion coefficients depend on the details of the dynamics of the theory and may depend logarithmically on the ratio Λ^2/m_h^2 and/or m_l/Λ. We perform an expansion in m_l/Λ since chiral perturbation theory works well for u and d quarks. The rationale for treating the strange quark differently than the light or heavy flavors is that we do not want to assume that either the heavy quark or chiral expansions converge well for the strange quark.

We define a fixed reference scale $\mu_0 > m_t$ with $\alpha(\mu_0) \equiv \alpha_0$. The scale Λ is defined by keeping $\alpha(\Lambda) \equiv \alpha_\Lambda$ fixed; the value itself is unimportant to us, only its logarithmic derivative matters. Holding gauge couplings α_0 and α_Λ fixed, we can invoke the Feynman-Hellman theorem to compute the $\bar{q}q$ matrix element in the nucleon by differentiating M with respect to $\bar{m}_q(\mu_0)$:

$$\bar{m}_q(\mu_0) \frac{\partial M}{\partial \bar{m}_q(\mu_0)} \bigg|_{\alpha_0, \alpha_\Lambda} = \bar{m}_q(\mu_0) \frac{\partial}{\partial \bar{m}_q(\mu_0)} \langle N | H_{QCD} | N \rangle$$
$$= \langle N | \bar{m}_q(\mu) \bar{q}q | N \rangle. \qquad (6)$$

The equality above follows from the Feynman-Hellman theorem since m_q enters the QCD Hamiltonian in the combination $m_q \bar{q}q$. We have not specified a renormalization scale in the matrix elements as they are renormalization scale invariant.

Let us first consider the chiral limit $m_u = m_d = m_s = 0$. Approximating (5) with

$$M = c_0 \Lambda + \mathcal{O}\left(\frac{\Lambda^3}{m_h^2} \right) \qquad (7)$$

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we can express the heavy matrix element as

$$\mathcal{M}_q = M\frac{\bar{m}_q(\mu_0)}{\Lambda}\frac{\partial\Lambda}{\partial\bar{m}_q(\mu_0)}\bigg|_{\alpha_0,\,\alpha_\Lambda}\left[1 + \mathcal{O}\left(\frac{\Lambda^3}{Mm_h^2}\right)\right]. \tag{8}$$

Thus to leading order in Λ^3/Mm_h^2 the heavy quark matrix element is determined by how Λ varies when we vary the "bare" quark mass $\bar{m}_q(\mu_0)$, keeping α_Λ and the "bare" gauge coupling α_0 fixed (see Fig. 1). This calculation requires knowledge of the anomalous mass dimension, the beta function for the running of the gauge coupling, and the matching functions that relate the gauge coupling and the running quark masses in the full n_f and the effective $(n_f - 1)$-flavor gauge theories. Here we use the notation and results compiled in the paper by Chetyrkin, Kuhn, and Steinhauser [7] to which we refer reader for the formulas we use and the references to the original results.

The coupling constant $a(\mu) \equiv \alpha^{(n_f)}(\mu)/\pi$ in the theory with n_f "active" flavors runs with the scale μ according to the equation

$$\mu^2\frac{da}{d\mu^2} = \beta^{(n_f)}(a), \tag{9}$$

where the QCD beta function,

$$\beta^{(n_f)}(a) = -\sum_{k\geq 0}\beta_k^{(n_f)}a^{k+2},$$

has been calculated to four loop order and the coefficients $\beta_k^{(n_f)}$ are known for $k = 0, 1, 2, 3$. The current quark mass $m_q^{(n_f)}(\mu)$ of the theory with n_f active flavors obeys

$$\mu^2\frac{d}{d\mu^2}\bar{m}_q^{(n_f)}(\mu) = \bar{m}_q^{(n_f)}(\mu)\gamma_m^{(n_f)}(a), \tag{10}$$

where the anomalous mass dimension,

$$\gamma_m^{(n_f)}(a) = -\sum_{k\geq 0}\gamma_{m,k}^{(n_f)}a^{k+1},$$

has been calculated to four loop order and the coefficients $\gamma_{m,k}^{(n_f)}$ are known for $k = 0, 1, 2, 3$. At a heavy quark threshold $\mu = m_h$ the couplings and the quark running masses in the "full" and the effective theories are related by the matching conditions

$$a^{(n_f-1)}(m_h) = a^{(n_f)}(m_h)\,\zeta_g^2(a^{(n_f)}(m_h)),$$
$$\bar{m}_q^{(n_f-1)}(m_h) = \bar{m}_q^{(n_f)}(m_h)\,\zeta_m(a^{(n_f)}(m_h)). \tag{11}$$

The functions ζ_g^2 and ζ_m have been computed in perturbation theory to three loop order. Here $a^{(n_f)}(m_h), \bar{m}_q^{(n_f)}(m_h)$ and $a^{(n_f-1)}(m_h), \bar{m}_q^{(n_f-1)}(m_h)$ are the coupling and quark masses evaluated just above and below the flavor threshold, respectively.

Equations (9) and (10) may be integrated to give

$$\ln[\frac{\mu_0^2}{\Lambda^2}] = \int_{a(\Lambda)}^{a(\mu_0)}\frac{da}{\beta^{(n_f)}(a)}, \tag{12}$$

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$$\ln[\frac{\bar{m}_h(\mu_0)}{m_h}] = \int_{a(m_h)}^{a(\mu_0)} da \frac{\gamma_m^{(n_f)}(a)}{\beta^{(n_f)}(a)}. \tag{13}$$

In order to obtain the desired quantity in (8) we differentiate (12) with respect to $\ln \bar{m}_q(\mu_0)$, keeping μ_0, α_0, and α_Λ fixed and taking into account the flavor threshold discontinuities. The result is

$$\frac{\partial \ln \Lambda}{\partial \ln \bar{m}_q(\mu_0)} = \sum_h \left(1 - \frac{\beta^{(n_f)}(a_{n_f h})}{\beta^{(n_f-1)}(\zeta_g^2(a_{n_f h})a_{n_f h})} \frac{d\,\zeta_g^2(a_{n_f h})a_{n_f h}}{da_{n_f h}} \right) \frac{\partial \ln m_h}{\partial \ln \bar{m}_q(\mu_0)} \tag{14}$$

Here $a_{n_f h} \equiv a^{(n_f)}(m_h)$ and the sum is over the heavy flavors lighter than the flavor q and the q itself. To calculate $\partial \ln m_h / \partial \ln \bar{m}_q(\mu_0)$ we differentiate (13) with respect to $\ln \bar{m}_q(\mu_0)$ keeping μ_0, α_0 fixed and taking into account the flavor threshold discontinuities in the running quark masses and coupling constant.

Now let us take into account the light quark masses. Working in the isospin limit $m_u = m_d = m_l$, we approximate (5) with

$$M = c_0 \Lambda + 2\,c_{10hl}\,m_l + \Delta_s M + \mathcal{O}\left(\frac{\Lambda^3}{m_h^2}, \frac{m_l^2}{\Lambda} \right) \tag{15}$$

and the heavy matrix element expression is modified to

$$\mathcal{M}_q = M\,(1 - x_{uds}) \frac{\bar{m}_q(\mu_0)}{\Lambda} \frac{\partial \Lambda}{\partial \bar{m}_q(\mu_0)} \bigg|_{\alpha_0, \alpha_\Lambda} \left[1 + \mathcal{O}\left(\frac{\Lambda^3}{Mm_h^2}, \frac{m_s^2}{M\,\Lambda} \right) \right], \tag{16}$$

where x_{uds} is defined in (2); $\Sigma_{\pi N} \simeq 45$ MeV is defined in (4) and is known from pion-nucleon scattering, while the value of the strange matrix element, $\mathcal{M}_s \simeq 200$ MeV, that enters $\Delta_s M$ from (3) is quite uncertain and varies by hundreds of MeV depending on the method of calculation [2,3].

We used the following values of parameters: $M_Z = 91.18$ GeV, $\alpha(M_Z) = 0.117$, $m_c = 1.2$ GeV, $m_b = 4.2$ GeV and $m_t = 175$ GeV [8]. The numerical results obtained are shown in (1).

III. CALCULATION IN THE THREE FLAVOR TOY MODEL

The value of $\mathcal{M}_s$ has been calculated in baryon chiral perturbation theory. The leading order calculation predicts linear growth of the matrix element as a function of m_s, however, at one loop order one gets sizable negative contributions from the meson loops proportional to $m_s^{3/2}$ [2,3]. This indicates that chiral perturbation theory may not be the right tool for the calculation. Calculations of $\mathcal{M}_s$ in the Skyrme model lead to the same conclusion [9]. This is unfortunate since the value of $\mathcal{M}_s$ may have important consequences for the nature of dense matter [1], and the fate of supernova remnants [10]. It could also be relevant for the properties of the strongly bound kaonic system K^-ppn observed recently [11].

One may, however, gain qualitative insight into this problem using a rather different approach. We calculate the strange matrix element in three flavor QCD with a hypothetical heavy strange quark (such that $m_s \gg \Lambda$) and massless u and d quarks using the technique of the previous section.
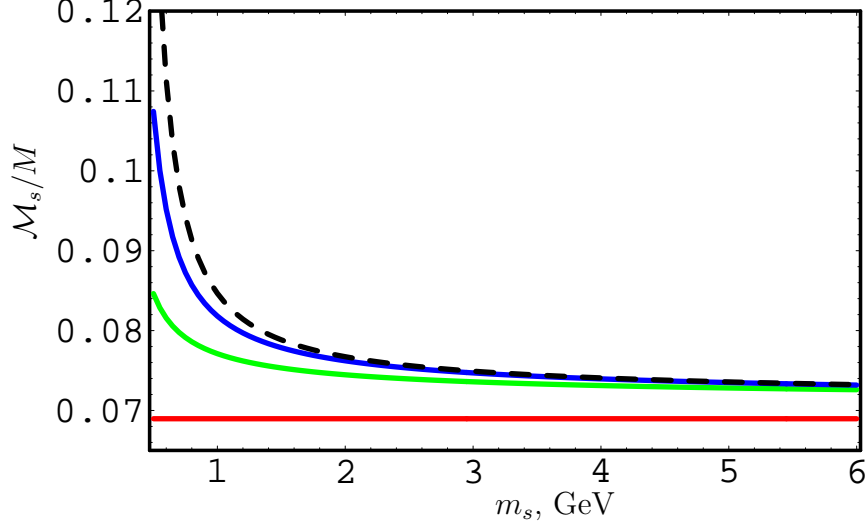
FIG. 2. Shown here is the perturbative result for $\mathcal{M}_s/M \equiv \langle N|m_s\bar{s}s|N\rangle/M$ versus m_s in GeV calculated in the toy three flavor model with massless u and d quarks and a heavy s quark. At the leading order $\mathcal{M}_s/M = 2/29 \simeq 0.069$ (the red line); the green line corresponds to the NLO result, the blue line - NNLO and the black dashed line - NNNLO. Perturbation theory is clearly breaking down for $m_s \leq 1$ GeV.

In the effective $n_f = 2$ theory the mass of the nucleon M is given by an expansion in m_s^{-2} of the form

$$M = \Lambda \times \sum_{n=0}^{\infty} c_n \left(\frac{\Lambda^2}{m_s^2}\right)^n ,\qquad(17)$$

where the coefficients c_n will depend logarithmically on the ratio Λ/m_s. Defining a reference scale $\mu_0 > m_s$ with $\bar{m}_s(\mu_0) \equiv m_{s0}$ and gauge coupling $\alpha(\mu_0) \equiv \alpha_0$, and using the same line of argument as in the previous section, we can express the strange matrix element as

$$\mathcal{M}_s = M \frac{m_{s0}}{\Lambda} \frac{\partial\Lambda}{\partial m_{s0}}\bigg|_{\alpha_0,\alpha_\Lambda} \left[1 + \mathcal{O}\left(\frac{\Lambda^3}{m_s^2 M}\right)\right] .\qquad(18)$$

Thus to leading order in Λ^3/Mm_s^2 the strange matrix element is determined by how Λ varies when we vary the "bare" strange quark mass, keeping α_Λ and the "bare" gauge coupling α_0 fixed.

In order to obtain the desired quantity in (18) we differentiate both sides of (12) and (13) with respect to $\ln m_{s0}$, keeping μ_0, α_0, and α_Λ fixed and taking into account the strange quark threshold discontinuity. We get

$$\begin{aligned}
\frac{1}{M}\langle N|m_s\bar{s}s|N\rangle &= \frac{\partial \ln \Lambda}{\partial \ln m_{s0}}\bigg|_{\alpha_0,\alpha_\Lambda} = \\
&= \frac{\partial \ln m_s}{\partial \ln m_{s0}} \left[1 - \frac{\beta^{(3)}(a)}{\beta^{(2)}(\zeta_g^2(a)\,a)} \frac{d}{da}(\zeta_g^2(a)\,a)\right] = \\
&= \left[\frac{1}{1-2\gamma_m^{(3)}(a)}\right] \left[1 - \frac{\beta^{(3)}(a)}{\beta^{(2)}(\zeta_g^2(a)\,a)} \frac{d}{da}(\zeta_g^2(a)\,a)\right],
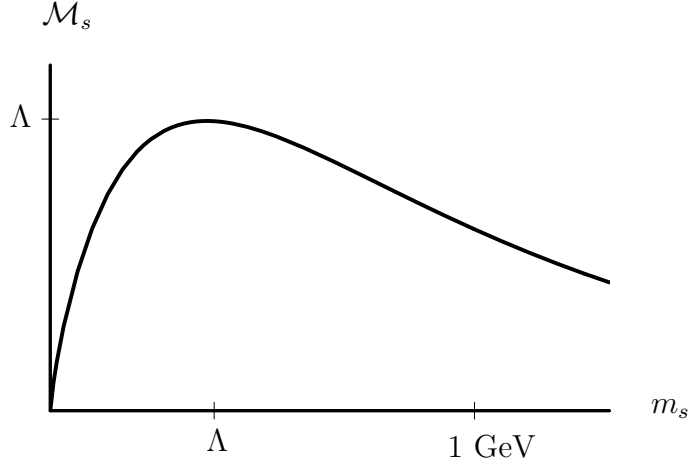\end{aligned}\qquad(19)$$

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FIG. 3.

where $a \equiv a^{(3)}(m_s)$. The coupling $a^{(3)}(m_s)$ is calculated by running down from $\mu = M_Z$ to $\mu = m_s$ using (12) with the real world initial condition $\alpha(M_Z) = 0.117$.

Fig. 2 shows $\mathcal{M}_s/M \equiv \langle N|m_s\bar{s}s|N\rangle/M$ as a function of m_s calculated through four loop order in perturbation theory. It indicates that QCD may favor a bigger value of the strange matrix element for the physical m_s. A reasonable guess for $\mathcal{M}_s(m_s)$ in the nonperturbative regime consistent with what we know about $\mathcal{M}_s$ for the values of m_s, both larger and smaller than its physical value, is sketched in Fig. 3.

Acknowledgments

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